DIFFERENTIAL-GEOMETRIC METHODS FOR CONTROL OF ELECTRIC MOTORS

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SUMMARY

The differential-geometric techniques of nonlinear control developed over the last 20 years or so include static and dynamic feedback linearization, input–output linearization, nonlinear state observers and disturbance decoupling. The theory has now reached a level of maturity where control practitioners are making effective use of the techniques for electric motors. Indeed, DC and AC motors have well-defined nonlinear mathematical models which often satisfy the structural conditions required of the differential-geometric theory. In this paper, the application of various differential-geometric methods of nonlinear control is shown by way of examples including DC motors (series, shunt and separately excited), induction motors, synchronous motors and DC–DC converters. A number of contributions are surveyed which show the benefits of the methods for the design of global control laws by systematic means. © 1998 John Wiley & Sons, Ltd.

Key words: electric motors; feedback linearization; nonlinear control

1. INTRODUCTION

Just as linear algebra is an essential mathematical tool to study the structure of linear control systems, the last 20 years have shown that differential geometry is an essential tool for the study of the structural properties of nonlinear control systems. In fact, one can consider the differential-geometric approach presented in References 1–3 as a natural generalization of Wonham’s classic work‡ on the structure of linear control systems. The intent of this paper is to show, by means of examples, that the approach has led to effective ways of designing control systems for electric motors. The usefulness of the techniques is that one can often find a global, linearizing control law, avoiding the need for gain scheduling or for slow operation. The development of the differential-geometric methodology has provided a formal systematic approach to develop these nonlinear control laws including techniques such as dynamic feedback linearization that, hitherto, had not been considered.

An observation of the paper is that, it is often the case that two or more of the techniques apply to a given nonlinear control system, e.g., a system may be both feedback linearizable and input–output linearizable. While a choice must be made of which technique to use, such issues as

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the complexity of the resulting controller, singularities in the controller, and stability of the zero

dynamics, are found to reduce the number of viable options. The nonlinear systems here include
DC motors (series, shunt and separately excited), induction motors, synchronous motors and
DC–DC converters. Though beyond the scope of this paper, the differential-geometric methods
are applicable to electro-mechanical systems in general. For example, these techniques are used to
control a simple magnetic levitation system in Reference 5. For another view of the control of
electric machines, the reader is referred to the interesting paper by Taylor.6

The paper is outlined as follows: Section 2 presents feedback linearization techniques for
permanent magnet synchronous motors. Synchronous machines with both uniform and
non-uniform airgaps are considered. Section 3 presents both input–output and dynamic feedback
linearization controllers for the induction motor. Section 4 considers a model of the separately
excited DC motor which is then specialized to the series and shunt connected DC motors.
Controllers and observers based on the differential-geometric techniques are then presented. In
Section 5, feedback linearization control of DC–DC converters is considered along with
a nonlinear observers to estimate the load on the converters. Finally, some concluding remarks
are presented in Section 6.

2. PM SYNCHRONOUS MOTORS

A two pole-pair permanent magnet (PM) synchronous motor can be constructed with two stator
phases 90° apart along with a permanent magnet rotor as shown in Figure 1. By choosing the
stator currents to be sinusoidal waves 90° apart in time, a rotating magnetic field is set-up in the
airgap of the machine. As a result, a torque is produced on the permanent magnet rotor due to
magnetic attraction.

A mathematical model for such a motor with \( n_p \) pole pairs is given by7

\[
v_{Sa} = R_i_{Sa} + \frac{d\lambda_{Sa}}{dt} \\
v_{Sb} = R_i_{Sb} + \frac{d\lambda_{Sb}}{dt}
\]

(1)

![Figure 1. PM synchronous motor](image)
The stator fluxes are given by

\[ \lambda_{Sa} = Li_{sa} + K_m \cos(n_p \theta) + L_g (i_{sa} \cos(2n_p \theta) + i_{sb} \sin(2n_p \theta)) \]
\[ \lambda_{Sb} = Li_{sb} + K_m \sin(n_p \theta) + L_g (i_{sa} \sin(2n_p \theta) + i_{sb} \cos(2n_p \theta)) \]

(2)

The third term is due to a non-uniform airgap, which is turn results in a variation of the reluctance across the airgap depending on the rotor position. If the airgap is uniform, then \( L_g = 0 \).

The second term \((K_m)\) is the flux in the stator phase due to the permanent magnet on the rotor.

The torque produced by the motor is given by

\[ \tau = 2n_p L_g ((i_{sa}^2 - i_{sb}^2) \sin(2n_p \theta) + 2i_{sa}i_{sb} \cos(2n_p \theta)) + n_p K_m (-i_{sa} \cos(n_p \theta) + i_{sb} \sin(n_p \theta)) \]

(3)

The first term on the right-hand side is the reluctance torque while the second term is the torque produced by the interaction of the rotor’s permanent magnet with the rotating magnetic field set-up by the stator currents. For most permanent magnet synchronous motors, \( L_g \approx 0 \), so that the motor torque is simply given by the second term. Variable reluctance motors do not have permanent magnet rotors so that \( K_m = 0 \), and torque is produced using the reluctance change due to a non-uniform airgap. Such motors are feedback linearizable with experimental results reported in References 9 and 10. Some synchronous motors such as the so-called brushless DC motor, exhibit both types of torque.

The model (1)–(3) is quite complicated, but a dramatic simplification occurs by use of the so-called dq transformation given by

\[
\begin{bmatrix}
  i_d \\
  i_q
\end{bmatrix} =
\begin{bmatrix}
  \cos(n_p \theta) & \sin(n_p \theta) \\
  -\sin(n_p \theta) & \cos(n_p \theta)
\end{bmatrix}
\begin{bmatrix}
  i_{sa} \\
  i_{sb}
\end{bmatrix}
\]

(4)

\[
\begin{bmatrix}
  v_d \\
  v_q
\end{bmatrix} =
\begin{bmatrix}
  \cos(n_p \theta) & \sin(n_p \theta) \\
  -\sin(n_p \theta) & \cos(n_p \theta)
\end{bmatrix}
\begin{bmatrix}
  v_{sa} \\
  v_{sb}
\end{bmatrix}
\]

(5)

As a result of the dq transformation, the model simplifies to

\[ L_d \frac{di_d}{dt} = -R i_d - n_p \omega L_q i_q + v_d \]
\[ L_q \frac{di_q}{dt} = -R i_q - n_p \omega L_d i_d - n_p \omega K_m + v_q \]
\[ J \frac{d\omega}{dt} = n_p (K_m i_q + (L_d - L_q) i_d) \]
\[ \frac{d\theta}{dt} = \omega \]

(6)

where \( L_d \triangleq L + L_q, L_q \triangleq L - L_g \).

2.1. PM synchronous motors with non-uniform airgaps

For a PM synchronous motor with a non-uniform airgap, the system is linearizable by feedback after transforming the model (6) to a new coordinate system specified by

\[ \theta = \theta \]
\[ \omega = \omega \]
In these coordinates, \((L_d - L_q = 2L_q)\), the model becomes

\[
\frac{d\theta}{dt} = \omega \\
\frac{d\omega}{dt} = \alpha \\
\frac{d\alpha}{dt} = f_1(\omega, i_d, i_q) + 2L_g n_p v_d/L_d + n_p (K_m + 2L_g i_d)v_q/L_q \\
\frac{di_d}{dt} = f_2(\omega, i_d, i_q) + v_d/L_d
\]

where

\[
f_1(\omega, i_d, i_q) = \frac{2L_g n_p}{L_d} (- R i_d - n_p \omega L_q i_q) + \frac{n_p}{L_q} (K_m + 2L_g i_d)(- R i_d - n_p \omega L_d i_q - n_p \omega K_m) \\
f_2(\omega, i_d, i_q) = (- R i_d - n_p \omega L_q i_q)/L_d
\]

Then the nonlinear feedback

\[
\begin{bmatrix}
  v_d \\
  v_q
\end{bmatrix} = \begin{bmatrix}
  \frac{2L_g n_p}{L_d} & \frac{n_p(K_m + 2L_g i_d)}{L_q} \\
  \frac{n_p(K_m + 2L_g i_d)}{L_q} & 0
\end{bmatrix}^{-1} \begin{bmatrix}
  -f_1 + u_d \\
  -f_2 + u_q
\end{bmatrix}
\]

results in a linear system. Typically, \(i_d \leq 0\) is maintained so the controller is nonsingular as long as \(i_d > -K_m/2L_g\). However, \(i_d = -K_m/2L_g\) would typically exceed the current rating of the motor so this singularity condition is not a problem in practice.\(^{11}\)

### 2.2. PM synchronous motor with uniform airgaps

In the case of a uniform airgap motor \(L_g = 0\), the model (1)–(3) leads to a simplified state-space model given by

\[
L \frac{di_{sa}}{dt} = - R i_a + K_m \omega \sin(n_p \theta) + v_{sa} \\
L \frac{di_{sb}}{dt} = - R i_b - K_m \omega \cos(n_p \theta) + v_{sb} \\
J \frac{d\omega}{dt} = - K_m i_{sa} \sin(n_p \theta) + K_m i_{sb} \cos(n_p \theta) \\
\frac{d\theta}{dt} = \omega
\]
As $L_q = 0$ results in $L_d = L_q = L$ or $L_d - L_q = 0$, inspection of (6) shows that the $dq$ transformation is a feedback linearizing transformation when the airgap is uniform. The derivation of (4) and (5) as the feedback linearizing transformation for (7) using the differential-geometric theory is given in Reference 12. As shown in Reference 13, a permanent magnet stepper (synchronous) motor is also accurately modeled by (7). (A cutaway view of a stepper motor is given in Figure 2.) With $L_d - L_q = 0$, the system model (6) is feedback linearized by choosing

$$v_d = -N_rL_\omega i_q + Lw_d$$

$$v_q = +N_rL_\omega i_d + Lw_q$$

Details on specifying $w_d$, $w_q$ as a linear feedback control law to force the motor to track a prespecified trajectory are given in Reference 13. As an example of the performance enhancements achievable using this feedback linearization controller, Figure 3 shows experimental results of a PM stepper motor being rotated $180^\circ$ in 30 ms. The corresponding speed tracking is shown in Figure 4.

![PM stepper motor](image)

**Figure 2. PM stepper motor**

![Graph](image)

**Figure 3. $\theta$ in rad vs time in s**
2.2.1. Full order speed observer. Typically, PM synchronous motors have position and current sensors, but the speed must be estimated to use it for feedback. A full-order state observer for the system (7) can be constructed to estimate the speed based on the theory of Krener and Respondek.

Specifically, as shown in Reference 15, consider the change of state coordinates

\[ x_1 = i_a + (K_m/n_p L) \cos(n_p \theta) \]
\[ x_2 = i_b + (K_m/n_p L) \sin(n_p \theta) \]
\[ x_3 = \omega \]
\[ x_4 = \theta \]

and change of output

\[ y_1 = i_a + (K_m/n_p L) \cos(n_p \theta) \]
\[ y_2 = i_b + (K_m/n_p L) \sin(n_p \theta) \]
\[ y_3 = \theta \]

so that (7) becomes

\[ \dot{x} = Ax + \varphi(y, v) \]
\[ y = Cx \]

where

\[ A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

\[ \varphi(y, v) = \begin{bmatrix} -(R/L)y_1 + (RK_m/n_p L^2) \cos(n_p y_3) + v_a \\ -(R/L)y_2 + (RK_m/n_p L^2) \sin(n_p y_3) + v_b \\ -(K_m/J)y_1 \sin(n_p y_3) + (K_m/J)y_2 \cos(n_p y_3) \end{bmatrix}. \]
As the pair \( \{ C, A \} \) is observable and \( \varphi(y, v) \) is known (measurable), it is therefore straightforward to construct an observer with linear error dynamics to estimate the complete state and in particular, the speed. Experimental results based on this approach are given in Reference 15.

**Remark**

Another important class of synchronous machines are synchronous generators. Marino\(^{16}\) has shown the applicability of feedback linearization to such machines.

### 3. INDUCTION MOTORS

As far as electric drives are concerned, the induction motor has been considered a kind of benchmark control problem. This is due to it having a quite complicated nonlinear mathematical model along with two of its variables (equivalent rotor currents/fluxes) not usually measured. In contrast to PM synchronous motors, induction motors are not feedback linearizable.\(^{17}\)

A breakthrough in terms of controlling the induction motor was made by Blaschke\(^{18}\) whose technique is now referred to as field-oriented control. In Marino et al.\(^{17}\) it was shown that field-oriented control can be viewed as an appropriate nonlinear change of coordinates of the system equations. Furthermore, the connection of field-oriented control to input–output linearization control\(^{19}\) is now well understood.\(^{20, 17, 21}\)

To summarize the differential-geometric approaches, consider a mathematical model of a sinusoidally wound, two-phase, \( n_p \) pole pair induction motor. Let \( i_{sa} \) and \( i_{sb} \) denote the currents in phases \( a \) and \( b \) of the stator, \( \psi_{Ra} \) and \( \psi_{Rb} \) denote the fluxes in phases \( a \) and \( b \) of the rotor, \( \theta \) be the rotor position, \( \omega \) be the rotor speed and, \( u_{sa} \) and \( u_{sb} \) be the applied voltage to phases \( a \) and \( b \) of the stator. These quantities are indicated in Figure 5 with \( n_p = 1 \) (adapted from Figure 4.1.7a of Reference 22) where the single coil in the figure for each phase represents a multi-turn sinusoidally wound phase.

![Figure 5. Induction motor](image-url)
To simplify the presentation, it is assumed the amplifiers have been configured for current command so that the phase currents $i_{Sa}, i_{Sb}$ can be considered as the inputs. This is usually accomplished by high-gain feedback of the currents to the voltage sources. In this case, the so-called $a-b$ model (or two-phase equivalent model) of the induction motor in terms of $\theta, \omega, \psi_{Ra}, \psi_{Rb}, i_{Sa}$ and $i_{Sb}^{18}$

$$\frac{d\theta}{dt} = \omega$$

$$\frac{d\omega}{dt} = -\frac{n_p M}{JL_R} (i_{Sb}\psi_{Ra} - i_{Sa}\psi_{Rb}) - \frac{\tau_L}{J}$$

$$\frac{d\psi_{Ra}}{dt} = - \frac{R_R}{L_R} \psi_{Ra} - n_p \omega \psi_{Rb} + \frac{M R_R}{L_R} i_{Sa}$$

$$\frac{d\psi_{Rb}}{dt} = - \frac{R_R}{L_R} \psi_{Ra} + n_p \omega \psi_{Rb} + \frac{M R_R}{L_R} i_{Sb}$$

(8)

where $\sigma \equiv 1 - M^2/L_S L_R$. In practice, the rotor fluxes $\psi_{Ra}, \psi_{Rb}$ (or, equivalently, the rotor currents $i_{Ra}, i_{Rb}$) are not available for feedback. Further, if the motor has a position sensor, then it typically does not have a speed sensor.

3.1. Field-oriented control

The state of the art in induction motor control is the so-called ‘field-oriented’ (or, vector control) technique introduced by Blaschke $^{18}$ As shown in Reference 24, one can interpret this approach by considering an equivalent model of (8) found using the following nonlinear state-space transformation

$$\theta = \theta$$

$$\omega = \omega$$

$$\rho = \tan^{-1}(\psi_{Rb}/\psi_{Ra})$$

$$\psi_d = \sqrt{\psi_{Ra}^2 + \psi_{Rb}^2}$$

$$i_d = + \cos(\rho) i_{Sa} + \sin(\rho) i_{Sb}$$

$$i_q = - \sin(\rho) i_{Sa} + \cos(\rho) i_{Sb}$$

(9)

to represent the system is the rotor-flux field-oriented coordinates.

Note that $\psi_d$ and $\rho$ are just the polar coordinates for the ordered pair $(\psi_{Ra}, \psi_{Rb})$. The term field-oriented refers to their new rotating coordinate system aligned with the field flux whose position is $\rho$ and magnitude is $\psi_d$. This transformation of the currents/fluxes/voltages is referred to as the direct-quadrature or dq transformation: Its principle and its result are similar to the dq transformation for synchronous motors, but it is a different transformation on the dynamic model.
In the new coordinates, the system equations become

\[
\begin{align*}
\frac{d\theta}{dt} &= \omega \\
\frac{d\omega}{dt} &= \mu \psi_d i_d - \frac{\tau_L}{J} \\
\frac{d\psi_d}{dt} &= -\eta \psi_d + \eta M i_d \\
\frac{d\rho}{dt} &= n_p \omega + \eta M i_q / \psi_d
\end{align*}
\]  

Consequently, the electromagnetic torque \( \tau = J \mu \psi_d i_d \) is proportional to the product of two-state variables. Field-oriented control consists in choosing the input \( i_d \) to keep \( \psi_d \) constant and then controlling the speed \( \omega \) using the input \( i_q \). In order to achieve high speeds with a limited voltage supply, it is often necessary to decrease the rotor flux \( \psi_d \) as a function of the speed \( \omega \). This is referred to as \textit{field weakening} and a standard flux reference to carry this out is given by

\[
\psi_{d\text{ref}} = \begin{cases} 
\psi_{d0} & \text{for } |\omega| \leq \omega_{\text{base}} \\
\psi_{d0} \frac{\omega_{\text{base}}}{\omega} & \text{for } |\omega| > \omega_{\text{base}}
\end{cases}
\]

where \( \psi_{d0} \) is the rated flux. Field-oriented control is based on the assumption that \( \psi_d \) is constant. To address the problem of simultaneously tracking the flux and speed, the methods of input–output linearization and dynamic feedback linearization have been proposed.

3.2. Input–output linearization

A quite effective way to achieve decoupling of the flux and speed dynamics is the so-called \textit{input–output linearizing} controller. \(24, 20, 25, 19, 21, 57\) Specifically, let \( i_d = u_1 \) and \( i_q = u_2 / (\mu \psi_d) \) so that the system (10) becomes

\[
\begin{align*}
\frac{d\omega}{dt} &= u_2 - \frac{\tau_L}{J} \\
\frac{d\psi_d}{dt} &= -\eta \psi_d + \eta M u_1 \\
\frac{d\rho}{dt} &= n_p \omega + \eta M i_q / \psi_d
\end{align*}
\]  

The flux dynamics are now decoupled from the speed dynamics. With \( u_1, u_2 \) the inputs and \( \omega, \psi_d \) the outputs, the system (11) is \textit{linear} from the new inputs to the outputs, i.e., \textit{input–output linearization} has been achieved. The dynamics of \( \rho \) are nonlinear and made unobservable. However, the boundedness of \( \rho \) is not an issue since it is an angle. Experimental results are reported in References 21 and 25 which exploit the method of input–output linearization. As an example, Figure 6 shows the speed tracking of a small fractional horsepower induction motor while simultaneously having the flux magnitude track a time-varying reference as shown in
Furthermore, this motion could not have been accomplished without varying the flux as shown in Figure 7.

3.3. Dynamic feedback linearization

It has been shown in Reference 24 that the induction motor is not feedback linearizable. By considering the speed as a constant (slowly varying) parameter, DeLuca and Ulivi have shown that the electrical dynamics of the motor are feedback linearizable. On the other hand, the constant speed requirement can be removed by considering dynamic feedback linearization. By adding an integrator to one of the inputs of the induction motor, the resulting (higher-order) system turns out to feedback linearizable. Recently, Martin and Rouchon have shown that the dynamic feedback linearizability of the induction motor can be simplified using the notion of flatness.
3.3.1. Addition of an integrator to the d-axis input. If only speed control is required, consider the addition of an integrator in the \( d \)-axis of (10) by letting

\[
\begin{align*}
    x_1 &= \omega \\
    x_2 &= \psi_d \\
    x_3 &= \rho \\
    x_4 &= i_d \\
    v_1 &= \frac{dx_4}{dt} \\
    v_2 &= i_q
\end{align*}
\]

Then (10) becomes

\[
\begin{align*}
    \frac{dx_1}{dt} &= \mu x_2 v_2 - \tau_L/J \\
    \frac{dx_2}{dt} &= -\eta x_2 + \eta M x_4 \\
    \frac{dx_3}{dt} &= n_p x_1 + \eta M v_2 / x_2 \\
    \frac{dx_4}{dt} &= v_1
\end{align*}
\]

Following Chiasson\(^{34}\) a feedback linearizing transformation is

\[
\begin{align*}
    z_1 &= x_2 \\
    z_2 &= -\eta x_2 + \eta M x_4 \\
    z_3 &= x_1 - \mu x_2^2 x_3 / \eta M \\
    z_4 &= \frac{2\mu}{M} x_2 x_3 (x_2 - M x_4) - \frac{\tau_L}{J} - \frac{\mu n_p}{\eta M} x_2^2 x_1
\end{align*}
\]

(12)

In these new co-ordinates, the system equations become

\[
\begin{align*}
    \frac{dz_1}{dt} &= z_2 \\
    \frac{dz_2}{dt} &= a_1(x) + b_{11}(x) v_1 + b_{12}(x) v_2 \\
    \frac{dz_3}{dt} &= z_4 \\
    \frac{dz_4}{dt} &= a_2(x) + b_{21}(x) v_1 + b_{22}(x) v_2
\end{align*}
\]
where

\[ a_1(x) = -\eta(-\eta x_2 + \eta M_4) \]

\[ a_2(x) = -2\eta M\mu x_3^2 + 6\eta \mu x_2 x_3 x_4 - 4\mu n_p x_1 x_2 x_4 + \frac{\mu n_p}{\eta M} \tau_1 x_2^2 \]

\[ + \frac{1}{M} (-4\eta \mu x_2^2 x_3 + 4\mu n_p x_1^2 x_2^2) \]

and

\[ b_{11}(x) = \eta M \]

\[ b_{12}(x) = -2\mu x_2 x_3 \]

\[ b_{21}(x) = 0 \]

\[ b_{22}(x) = -2\mu (-\eta x_2 + \eta M x_4) - \frac{\mu n_p}{\eta M} x_2^3 \]

Application of the feedback

\[
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}^{-1} \begin{bmatrix} u_1 - a_1(x) \\ u_2 - a_2(x) \end{bmatrix}
\]

results in two decoupled second-order linear systems

\[
\begin{align*}
dz_1/dt &= z_2 \\
dz_2/dt &= u_1 \\
dz_3/dt &= z_4 \\
dz_4/dt &= u_2
\end{align*}
\]

(13)

The controller is singular when

\[ \det \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = 0 \]

which, in terms of the dq co-ordinates, is

\[ -(2\eta M \mu) d\psi_d/dt - \mu^2 n_p \psi_d^3 = 0 \]

This can be easily avoided by choosing, for example, the input \( u_1 \) to regulate \( \psi_d \) to a positive constant (so that \( d\psi_d/dt = 0 \)). On the other hand, during field-weakening where \( \psi_d \) is reduced, this condition restricts how fast \( \psi_d \) can be decreased. At start-up of the motor, the singularity is not encountered as \( d\psi_d/dt > 0 \).

Remarks

Note from (12) that the speed \( \omega \) is not one of the transformed (new) state variables \( z_i \). The transformed state \( z \) is not directly measurable, but instead is computed through (12). As a result, the tracking accuracy of the speed is subject to any uncertainty in the parameters as well as any uncertainty in the estimation of the dq variables which are used to compute the state \( z \).
Furthermore, the speed $\omega$ is not linearly related to the $z_i$ so that $d\theta/dt = \omega$ is not a linear function of the $z_i$. Therefore, the position cannot be appended to the system (13) without losing linearity of the system.

### 3.3.2. Addition of an integrator to the $q$-axis input.

It is quite natural to consider adding an integrator in the $q$-axis of (10) rather than the $d$-axis for comparison. Let

- $x_1 = \omega$
- $x_2 = \psi_d$
- $x_3 = \rho$
- $x_4 = i_q$
- $v_1 \triangleq i_d$
- $v_2 \triangleq dx_4/dt$

Then (10) becomes

\[
\begin{align*}
\frac{dx_1}{dt} &= \mu x_2 v_2 - \tau_1/J \\
\frac{dx_2}{dt} &= -\eta x_2 + \eta M v_1 \\
\frac{dx_3}{dt} &= n_p x_1 + \eta M x_4/x_2 \\
\frac{dx_4}{dt} &= v_2
\end{align*}
\]

(14)

A feedback linearizing transformation is

- $z_1 = x_1$
- $z_2 = \mu x_2 x_4 - \tau_1/J$
- $z_3 = x_3$
- $z_4 = n_p x_1 + \eta M x_4/x_2$

(15)

The system equations are

\[
\begin{align*}
\frac{dz_1}{dt} &= z_2 \\
\frac{dz_2}{dt} &= a_1(x) + b_{11}(x)v_1 + b_{12}(x)v_2 \\
\frac{dz_3}{dt} &= z_4 \\
\frac{dz_4}{dt} &= a_2(x) + b_{21}(x)v_1 + b_{22}(x)v_2
\end{align*}
\]
where

\[ a_1(x) = - \eta x_2 x_4 \]
\[ a_2(x) = \mu \eta x_2 x_4 + \eta^2 M \frac{x_4}{x_2} - \frac{n_p \tau_L}{J} \]

and

\[ b_{11}(x) = \eta M \mu x_4 \]
\[ b_{12}(x) = \mu x_2 \]
\[ b_{21}(x) = - \eta^2 M^2 x_4/x_2^2 \]
\[ b_{22}(x) = \eta M/x_2 \]

Application of the feedback

\[
\begin{bmatrix}
  v_1 \\
  v_2
\end{bmatrix} = \begin{bmatrix}
  b_{11} & b_{12} \\
  b_{21} & b_{22}
\end{bmatrix}^{-1}
\begin{bmatrix}
  u_1 - a_1(x) \\
  u_2 - a_2(x)
\end{bmatrix}
\]

again results in two decoupled second-order systems as in the \(d\)-axis case. This controller is singular when

\[
\text{det}\begin{bmatrix}
  b_{11} & b_{12} \\
  b_{21} & b_{22}
\end{bmatrix} = 2\eta^2 M^2 \mu \frac{i_q}{\psi_d} = 0
\]

The singularity condition essentially requires that the torque \(\tau = J \mu \psi_d i_q = J \mu x_2 x_4 \neq 0\). A drawback here compared to the \(d\)-axis approach is the requirement that the quadrature current \(i_q\) be non-zero \((\psi_d > 0)\) is straightforward to maintain and is common to other controllers.\(^{18,19,24}\)

Adding the state variable \(z_0 = \theta\) to the transformation (15) gives a feedback linearizing controller for position control. However, to avoid the singularity of the controller, it is still required that \(i_q \neq 0\).

### 3.4. Flux and load-torque observers

The controllers discussed so far require full state feedback as well as knowledge of the load–torque \(\tau_L\). This is a problem as usually neither of them is completely available. Sometimes the rotor angle \(\theta\) is measured, without a separate sensor for the speed \(\omega\). The problem is then as follows: with only measurements of the stator currents and rotor position, estimate the fluxes \((\psi_{Ra}, \psi_{Rb})\), the rotor speed \(\omega\) and the load–torque \(\tau_L\).

As in the case of the control problem, the key to developing such an estimator is to find the appropriate coordinate system to do the design and analysis.\(^{35}\) In this case, the rotor coordinate system is the appropriate one. Specifically, as in Reference 36 consider the transformation

\[
\begin{align*}
  i_{sc} &\triangleq \cos(n_p \theta) i_{sa} + \sin(n_p \theta) i_{sb} \\
  i_{sy} &\triangleq - \sin(n_p \theta) i_{sa} + \cos(n_p \theta) i_{sb} \\
  \psi_{Rx} &\triangleq \cos(n_p \theta) \psi_{Ra} + \sin(n_p \theta) \psi_{Rb} \\
  \psi_{Ry} &\triangleq - \sin(n_p \theta) \psi_{Ra} + \cos(n_p \theta) \psi_{Rb}
\end{align*}
\]
In order to estimate the load–torque, it is modelled as a constant so that its dynamic equation is simply \( \frac{d(\tau_L/J)}{dt} = 0 \). Transforming the flux, speed and position dynamics of (8) by (16), and appending the load–torque equation, one obtains

\[
\begin{align*}
\frac{d\psi_{Rx}}{dt} &= -\frac{R_R}{L_R} \psi_{Rx} + \frac{MR_R}{L_R} i_{sx} \\
\frac{d\psi_{Ry}}{dt} &= -\frac{R_R}{L_R} \psi_{Ry} + \frac{MR_R}{L_R} i_{sy} \\
\frac{d\theta}{dt} &= \omega \\
\frac{d\omega}{dt} &= \mu(i_{sy}\psi_{Rx} - i_{sx}\psi_{Ry}) - \tau_L/J \\
\frac{d(\tau_L/J)}{dt} &= 0
\end{align*}
\]

The point here is that the (unmeasured) speed \( \omega \) eliminated from the flux equations with the consequence that an estimator for the fluxes, speed and load–torque is

\[
\begin{align*}
\frac{d\hat{\psi}_{Rx}}{dt} &= -\frac{R_R}{L_R} \hat{\psi}_{Rx} + \frac{MR_R}{L_R} i_{sx} \\
\frac{d\hat{\psi}_{Ry}}{dt} &= -\frac{R_R}{L_R} \hat{\psi}_{Ry} + \frac{MR_R}{L_R} i_{sy} \\
\frac{d\hat{\theta}}{dt} &= \hat{\omega} + l_1(\theta - \hat{\theta}) \\
\frac{d\hat{\omega}}{dt} &= \mu(i_{sy}\hat{\psi}_{Rx} - i_{sx}\hat{\psi}_{Ry}) - \hat{\tau}_L/J + l_2(\theta - \hat{\theta}) \\
\frac{d(\hat{\tau}_L/J)}{dt} &= 0 + l_3(\theta - \hat{\theta})
\end{align*}
\]

Subtracting (18) from (17) results in the error system

\[
\begin{align*}
\frac{de_{Rx}}{dt} &= \frac{R_R}{L_R} e_{Rx} \\
\frac{de_{Ry}}{dt} &= \frac{R_R}{L_R} e_{Ry} \\
\frac{de_{\theta}}{dt} &= e_{\theta} - l_1(\theta - \hat{\theta}) \\
\frac{de_{\omega}}{dt} &= \mu(i_{sy}e_{Rx} - i_{sx}e_{Ry}) - e_{\omega} - l_2(\theta - \hat{\theta}) \\
\frac{de_{\tau}}{dt} &= 0 - l_3(\theta - \hat{\theta})
\end{align*}
\]
The flux error is stable as $e_{Rx}(t) = e_{Rx}(0)e^{-t/T_x}$, $e_{Ry}(t) = e_{Ry}(0)e^{-t/T_y}$ go to 0 as $t \to \infty$. As long as the currents $i_{sx}, i_{sy}$ are bounded (consistent with the assumed current-command operation), the perturbation term $\mu(i_{sy}e_{Rx} - i_{sx}e_{Ry})$ due to the error in the estimate of motor torque dies out, that is, $\|i_{sx}e_{Rx} - i_{sx}e_{Rx}\| \leq Ce^{-t/T_x}$ for some $C > 0$. By choosing the gains $l_1, l_2$ and $l_3$ appropriately, the subsystem consisting of $e_\theta, e_\omega$ and $e_z$ is a stable linear time-invariant system driven by the exponentially decaying input $\mu(i_{sy}e_{Rx} - i_{sx}e_{Ry})$. Consequently, $e_\theta \to 0$, $e_\omega \to 0$, $e_z \to 0$ as $t \to \infty$, showing convergence of the estimator.

An approach for estimating the rotor fluxes with adaptation to the rotor time constant has recently been reported in Marino et al.\cite{Marino} Approaches for designing flux observers where the rate of convergence can be arbitrarily specified (assuming the speed is known) is given by Verghese and Sanders\cite{Verghese} with an even more flexible approach recently discovered by Martin and Rouchon\cite{Martin}.

3.5. Summary of differential-geometric techniques for induction motors

The dynamic feedback linearization controllers presented above can provide decoupled control of flux and speed. The speed $\omega$ can be made to track an arbitrary speed reference while the flux $\psi_d$ is varied for, say, field weakening. However, the $d$-axis controller cannot be used for position control nor is speed a transformed variable marking the speed tracking accuracy subject to uncertainties in both the motor parameters and the estimates of the $dq$ state variables. While the $q$-axis controller can provide position and speed tracking control, it requires that the quadrature current be non-zero to avoid singularities.

On the other hand, the input–output linearization controller not only provides decoupled control of speed and flux, it is also simpler and less sensitive to motor parameter uncertainty than the dynamic feedback linearization controllers. The induction motor is an example of a nonlinear control system for which several methods are applicable with widely different operational characteristics. Practical issues narrow down the choice and give a significant advantage to input–output linearization.

4. DIRECT CURRENT MOTORS

Control algorithms for DC motors based on differential-geometric methods have been given in Reference 2, 39, 41 and 48. To illustrate such methods, consider the simplified diagram of a DC motor shown in Figure 8. The windings on the rotor are referred to as the armature circuit while the windings on the stator are referred to as the field circuit. In the case of a permanent magnet DC motor, the field circuit is replaced by a permanent magnet.

While DC motors are normally considered to be easier to control than synchronous or induction motors, we consider here more complicated models accounting for magnetic saturation, and show how these can be treated by the theory.

4.1. Mathematical Model

In Figure 8, the armature circuit has terminals $T_1$ and $T_2$, and the field circuit has terminals $T'_1$ and $T'_2$. A mathematical model of the separately excited DC motor, i.e., one in which the field and armature circuits have independent voltage sources, is given
Here, \( i_a \) is the current in the armature circuit, \( \phi_t \) is the flux (linkage) in the field windings (and in the airgap assuming no leakage) and \( \omega \) is the motor’s angular speed. The (input) armature voltage \( V_a \) is applied between terminals \( T_1 \) and \( T_2 \) while the (input) field voltage \( V_f \) is applied between terminals \( T_1' \) and \( T_2' \). The field flux \( \phi_t \) is related to the field current \( i_t \) by \( \phi_t(i_t) = f(i_t) \) where \( f(\cdot) \) is the magnetization curve as shown in Figure 9. \( R_a \) and \( R_f \) denote the resistance of the armature and field windings, respectively. The armature inductance is denoted by \( L_a \) and the torque/back-emf constant by \( K_m \).

The magnetization curve is strictly increasing, symmetric with respect to origin and satisfies \( f(i) > 0 \) for \( i \neq 0 \). For \( i_t \) small, \( \phi_t(i_t) \) may be modelled as a linear function of \( i_t \), i.e. \( \phi_t(i_t) = L_t i_t \) where \( L_t = \phi_t(0) \).

Notice that in the first equation of the system (20), the back-emf (voltage) \( - K_m \phi_t \omega \) increases in proportion to the speed. This requires the input voltage \( V_a \) be at least large enough to overcome the back-emf to maintain the armature current. As in the case of the induction motor,
to have the motor achieve higher speeds within the voltage limits, field weakening is employed. This is accomplished by forcing $\phi_t$ to track a flux reference given by

$$\phi_{\text{ref}} = \frac{\phi_{t0}}{\omega} \quad \text{for } |\omega| \leq \omega_{\text{base}}$$

$$= \phi_{t0} \frac{\omega_{\text{base}}}{\omega} \quad \text{for } |\omega| \geq \omega_{\text{base}}$$

Field weakening results in a constant back-emf of $\phi_{t0}/\omega_{\text{base}}$ for speed greater than $\omega_{\text{base}}$. The trade-off is that the torque $K_m\phi_t i_a$ is less for the same $i_a$ due to the decrease in field flux.

4.2. Separately excited DC motor

In a separately excited DC motor, the input $V_t$, $V_a$ can be used as independent inputs. If the input $V_t$ is used to maintain $\phi_t$ (and thus $i_t$) constant, then the system dynamics (20) are linear. (In a permanent magnet DC machine, the field windings with terminals $T_1$ and $T_2$ along with the iron core are replaced with a permanent magnet resulting in $\phi_t = \text{constant}$. In this special case where $\phi_t$ is held constant, the dynamics of the motor are described by the first and third equation of the system (20). This is the well-known linear DC machine analysed in elementary control textbook and is not considered here. The concern here is the control problem when $\phi_t$ is not constant.

4.2.1. Armature control of a separately excited DC motor. With $x_1 = i_a$, $x_2 = \phi_t$, $x_3 = \omega$, $u_1 = V_a/L_a$ and $u_2 = V_t$, the model (20) becomes

$$\frac{d x_1}{dt} = -\frac{R_a}{L_a} x_1 - \frac{K_m}{L_a} x_2 x_3 + u_1$$

$$\frac{d x_2}{dt} = -R_f f^{-1}(x_2) + u_2$$

$$\frac{d x_3}{dt} = \frac{K_m}{J} x_2 x_1 - \frac{\tau_l}{J}$$

Consider the nonlinear transformation

$$z_1 = x_3$$

$$z_2 = (K_m/J) x_2 x_1$$

$$z_3 = x_2$$

so that in the new coordinates, the system (22) becomes

$$\frac{d z_1}{dt} = z_2 - \frac{\tau_l}{J}$$

$$\frac{d z_2}{dt} = g(x) + u_1 (K_m/J) x_2 + (K_m/J) x_1 u_2$$

$$\frac{d z_3}{dt} = -R_f f^{-1}(x_2) + u_2$$
where
\[ g(x) = \frac{K_m}{J} x_1 (R_f f^{-1}(x_2)) + \frac{K_m}{J} x_2 \left( -\frac{R_a}{L_a} x_1 - \frac{K_m}{L_a} x_2 x_3 \right) \]

Upon application of the feedback
\[
\begin{bmatrix}
  u_1 \\
  u_2
\end{bmatrix} = \begin{bmatrix}
  (K_m/J) x_2 & (K_m/J) x_1 & 0 \\
  0 & 1 & 0
\end{bmatrix}^{-1} \begin{bmatrix}
  -g(x) \\
  -R_f f^{-1}(x_2)
\end{bmatrix} + \begin{bmatrix}
  v_1 \\
  v_2
\end{bmatrix}
\]
results in the linear system
\[
\begin{align*}
  \frac{dz_1}{dt} &= z_2 - \tau_L/J \\
  \frac{dz_2}{dt} &= v_1 \\
  \frac{dz_3}{dt} &= v_2
\end{align*}
\]

The feedback is singular only if \( x_2 = \phi_f = 0 \). However, one may choose the input \( v_2 = V_f \) to ensure this does not occur. The importance of this approach is that the control of the flux through the input \( v_2 \). Finally, if position control is desired, with \( z_0 = \theta \), one simply appends \( dz_0/dt = z_1 \) to the system (23). The reader may be interested in comparing this differential geometric approach with a more conventional gain scheduling approach recently reported in Reference 42.

4.2.2. Field control of a separately excited DC motor. Another approach to controlling a separately excited DC motor is the so-called method of field control. In this approach the armature voltage is held constant while the field voltage is varied to control the speed. This is often used in large DC drives (e.g., rolling mills) since the armature circuit can require 2000–3000 A and 250 V while the field circuit may require only 20 A. Consequently, it is of great economic advantage to use an amplifier that only has to handle the 20 A of the field circuit together with a large generator at constant voltage for the armature circuit.

Using (20), the model for such a configuration is
\[
L_a \frac{di_a}{dt} = V_{a0} - R_a i_a - K_m \phi_f \omega
\]
\[
\frac{d\phi_f}{dt} = -R_f f^{-1}(\phi_f) + V_f
\]
\[
J \frac{d\omega}{dt} = K_m \phi_f i_a - \tau_L
\]

where \( V_{a0} \) is constant (typically, the output of a generator). The methods of feedback and input–output linearization controller for this system have been considered in detail by Isidori. As shown there the system is feedback linearizable with the transformation based on taking \( z_1 = Li_a^2 + J\omega^2 \) and computing its first and second derivatives. It turns out that the speed \( \omega \) is not
one of the transformed variables $z_i$, making this approach to speed control sensitive to uncertainty in the motor parameters $J$ and $L_a$.

Isidori\textsuperscript{2} has also considered speed control based on input–output linearization around the equilibrium points of (24). It turns out that (24) has two different equilibrium points for any given speed. The armature currents at these equilibrium points are:\textsuperscript{2}

$$i_{a0-\text{eq}1} = \frac{V_{a0} - \sqrt{V_{a0}^2 - 4R_s\omega_0\tau_L}}{2R_a}$$

$$i_{a0-\text{eq}2} = \frac{V_{a0} + \sqrt{V_{a0}^2 - 4R_s\omega_0\tau_L}}{2R_a}$$

Interestingly, with regard to input–output linearization with speed as the output, $i_{a0-\text{eq}1}$ is the current at the equilibrium point where the zero dynamics are unstable, while $i_{a0-\text{eq}2}$ is the current at the equilibrium point that has stable zero dynamics.\textsuperscript{2} However, the case with the stable zero dynamics requires a current $i_{a0-\text{eq}2}$ that is typically well beyond the rating of the motor.\textsuperscript{43} As a result, the input–output linearization controller is not practical and the feedback linearization algorithm proposed in Reference 2 is the only viable choice.

4.3. Series connected DC motor

A DC motor in which the field circuit is connected in series with the armature circuit is referred to as a Series DC motor. This connection is typically used in applications that require high torque at low speed such as traction drives (e.g., subway trains).\textsuperscript{18}

4.3.1. Control of the series DC motor. The separately excited motor model (20) is easily modified to obtain the model of the series connected motor. Specifically, the terminal $T_2'$ is connected to the terminal $T_1$ with the voltage $V$ applied between terminals $T_1'$ and $T_2$. This connection is shown in the equivalent circuit of the series DC motor shown in Figure 10. With the switch in Figure 10 open, the equations describing a series connected DC motor are

$$\frac{d(L_a i + \phi_f(i))}{dt} = -(R_f + R_a)i - K_m\phi_f(i)\omega + V$$

$$J \frac{d\omega}{dt} = K_m\phi_f(i)i - \tau_L$$

In this case, a feedback linearizing transformation is given by

$$z_1 = \omega$$

$$z_2 = (K_m/J)\phi_f(i)i - \tau_L/J$$

so that

$$\frac{dz_1}{dr} = z_2$$

$$\frac{dz_2}{dr} = \frac{K_m(\phi_f(i) + \phi_f(i))}{J(L_a + \phi_f(i))} \left( -(R_f + R_a)i - K_m\phi_f(i)\omega + V \right)$$
where $\phi'_i(i) \triangleq d\phi_i(i)/di$. The feedback

$$V = ((R_t + R_a)i + K_m\phi_t(i)\omega + u) \frac{J}{K_m (\phi'_t(i)i + \phi_t(i))}$$

results in the linear system

$$\frac{dz_1}{dt} = z_2$$

$$\frac{dz_2}{dt} = u$$

The feedback is singular only when $\phi'_t(i)i + \phi_t(i) = 0$, or equivalently, when $i = 0$. As in the case of the separately excited machine, this singularity is due to the physics of the machine, i.e., the machine cannot produce torque if the field flux is zero.

Series DC motors usually employ field weakening to avoid voltage supply limits at higher speeds. This is done by placing a switch along with a resistance $R_p$ in parallel with the field circuit as shown in Figure 10. When the switch is closed, the current in the field winding is reduced with a consequent decrease in the field flux and thus the back-emf.

The equations describing a series-wound DC motor with the switch closed are:

$$L_e \frac{di_a}{dt} = V - R_a i_a - R_p (i_a - i_t) - K_m \phi_t \omega$$

$$\frac{d\phi_t}{dt} = -R_t i_t + R_p (i_a - i_t)$$

$$J \frac{d\omega}{dt} = K_m \phi_t i_a - \tau_L$$
where \( i_t = f^{-1}(\phi_t) \). With \( \psi(\phi_t) = f^{-1}(\phi_t) \), a feedback linearizing transformation for this system is

\[
\begin{align*}
    x_1 &= \omega - \frac{K_m}{2R_pJ} \phi_t^2 \\
    x_2 &= \frac{K_m(R_f + R_p)}{JR_p} \phi_t \psi(\phi_t) \\
    x_3 &= \frac{K_m(R_f + R_p)}{JR_p} \left( \psi(\phi_t) + \phi_t \psi'(\phi_t) \right) \left( -R_f + R_p \right) \psi(\phi_t) + R_p i_a
\end{align*}
\]

resulting in the system of the form

\[
\frac{d}{dt} x_1 = x_2 \\
\frac{d}{dt} x_2 = x_3 \\
\frac{d}{dt} x_3 = a(i_a, \phi_t, \omega) + b(\phi_t)V
\]

where it can be shown that \( b(\phi_t) = 0 \) if and only if \( \phi_t = 0 \), or equivalently, if \( i_t = 0.44 \). However, speed is not one of the transformed state variables \( x_j \) in (27) complicating the design of the speed tracking as well as its tracking accuracy being subject to uncertainties in the motor parameters and flux computation.

The problems with the feedback linearization controller can be circumvented by considering input–output linearization. To do so, note that the model can be simplified as the armature inductance \( L_a \) in a typical DC motor is negligible. Setting \( L_a = 0 \) in the first equation of (26) and solving for \( i_a \) gives

\[
    i_a = \frac{(V + R_p i_t - K_m \phi_t \omega)}{R_a + R_p}
\]

Substituting this expression for \( i_a \) into (26) results in the reduced-order model

\[
\begin{align*}
    \frac{d}{dt} \phi_t &= -\left( R_f + R_p \right) i_t + \frac{R_p}{R_a + R_p} (V + R_p i_t - K_m \phi_t \omega) \\
    \frac{d}{dt} \omega &= \frac{\left( K_m/J \right) \phi_t}{R_a + R_p} (V + R_p i_t - K_m \phi_t \omega) - \frac{\tau_1}{J}
\end{align*}
\]

An input–output linearization controller is

\[
V = -R_p i_t + K_m \phi_t \omega + u \frac{R_a + R_p}{(K_m/J) \phi_t}
\]
so that (29) becomes
\[
\frac{d\phi_t}{dt} = -(R_t + R_p)f^{-1}(\phi_t) + \frac{R_p}{(K_m/J)\phi_t} u
\]
(30)
\[
\frac{d\omega}{dt} = u - \tau_L/J
\]

The speed $\omega$ is straightforwardly controlled using the input $u$. However, the zero dynamics described by the first equation in (30) must be checked for stability. To do so, multiply both sides by $\phi_t$ to obtain
\[
\frac{1}{2} \frac{d\phi_t^2}{dt} = -(R_t + R_p)\phi_t f^{-1}(\phi_t) + \frac{R_p}{(K_m/J)\phi_t} u
\]

Using the fact that $\phi_t f^{-1}(\phi_t) = f(i_t)\dot{i}_t$ is positive definite, it easily seen that this is BIBO stable. As the zero dynamics are stable, the input–output controller has advantages over the feedback linearization controller due to its simplicity as well as it being less sensitive to parameter uncertainty.

4.3.2. Speed and load-torque observer. Using the differential geometric methods pioneered in References 14, 45, 46 and 47 a non linear speed and load–torque observer with linear error dynamics can be constructed for the series DC motor. To do so, the first two equations of (26) are added (with $L_a = 0$) to get
\[
\frac{d\phi_t}{dt} = -R_i l_t - R_i j_a - K_m \phi_t \omega + V
\]
\[
J \frac{d\omega}{dt} = K_m \phi_t j_a - \tau_L
\]
(31)

Modelling the load–torque as a constant, the equation
\[
\frac{d(\tau_L/J)}{dt} = 0
\]
(32)
is appended to (31) to model the load–torque dynamics. With the nonlinear change of coordinates,\textsuperscript{44}
\[
z_1 = \ln(\phi_t)
\]
\[
z_2 = \omega
\]
\[
z_3 = \tau_L/J
\]
the system (31) (32) becomes
\[
\dot{z} = Az + \phi(i_a, i_t, \phi_t, V)
\]
\[
y = cz
\]

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where

\[
\begin{bmatrix}
  z_1 \\
  z_2 \\
  z_3
\end{bmatrix}
= \begin{bmatrix}
  -K_m & 0 \\
  0 & -1 \\
  0 & 0
\end{bmatrix}
\begin{bmatrix}
  z_1 \\
  z_2 \\
  z_3
\end{bmatrix}
+ \begin{bmatrix}
  0 \\
  0 \\
  0
\end{bmatrix}
\]

\[
\phi(i_a, i_f, \phi_f, V) = \begin{bmatrix}
  (-R_f i_f - R_a i_a + V)/\phi_f \\
  (K_m/J)\phi_f i_a \\
  0
\end{bmatrix}
\]

Defining an observer as

\[
d\hat{z}/dt = A\hat{z} + \phi(i_a, i_f, \phi_f, V) + l(y - \hat{y})
\]

\[
\hat{y} = c\hat{z}
\]

it is easily seen that the poles of the error system (described by the difference between (33) and (34)) can be arbitrarily assigned by proper choice of the gain vector \( l = [l_1 \ l_2 \ l_3]^T \).

In summary, based only on measurements of the field and armature currents, both the speed and load-torque can be estimated. Experimental results of the implementation of these differential-geometric techniques are given in Reference 48. In particular, Figure 11 shows the open-loop speed estimate \( \hat{z}_2 = \hat{\omega} \) using (34) along with the measured speed \( \omega \) vs. time.\(^{48}\)

### 4.4. Shunt connected DC machine

A shunt connected DC motor is one in which the armature and field circuit are connected in parallel. Specifically, in Figure 8, one ties together terminals \( T_1 \) and \( T'_1 \) as well as the terminals \( T_2 \) and \( T'_2 \) resulting in the equivalent circuit shown in Figure 12. The mathematical model is
Figure 12. Equivalent circuit for a shunt DC motor

easily found from (20) to be

\[
L_a \frac{di_a}{dt} = -R_s i_a - K_m \phi_f \omega + V
\]

\[
\frac{d\phi_f}{dt} = -R_f^{-1}(\phi_f) + V
\]

\[
J \frac{d\omega}{dt} = K_m \phi_f i_a - \tau_L
\]

\[
\frac{d\theta}{dt} = \omega
\]

where \( R = R_f + R_{adj} \) with \( R_{adj} \) an adjustable resistance added to the field circuit (see Figure 12). As in the case of the series DC motor, the armature inductance is negligible so that setting \( L_a = 0 \) and \( i_a = (V - K_m \phi_f \omega)/R_a \) results in the reduced-order system

\[
\frac{d\phi_f}{dt} = -R_f^{-1}(\phi_f) + V
\]

\[
\frac{d\omega}{dt} = -\frac{K_m}{JR_a} \phi_f^2 \omega + \frac{K_m}{JR_a} \phi_f V - \frac{\tau_L}{J}
\]

\[
\frac{d\theta}{dt} = \omega
\]

It turns out that this third-order system is not feedback linearizable. However, if speed control is considered rather than position control, then the dimension of the state-space model (35) is reduced from three to two. As a result, the system is feedback linearizable by the
transformation

\[ z_1 = \omega - \frac{K_m}{2JR_a} \phi_t^2 \]
\[ z_2 = \frac{dz_1}{dt} \]
\[ = -\frac{K_m^2}{JR_a} \phi_t^2 \omega + \frac{RK_m}{JR_a} \phi_t \psi(\phi_t) - \frac{\tau_t}{J} \]

where \( \psi(\phi_t) \triangleq f^{-1}(\phi_t) = i_t \). In these coordinates, the system dynamics have the form

\[ \frac{dz_1}{dt} = z_2 \]
\[ \frac{dz_2}{dt} = \alpha(\omega, \phi_t) + \beta(\omega, \phi_t) V \]

which is feedback linearizable. The expressions for \( \alpha(\omega, \phi_t), \beta(\omega, \phi_t) \) are somewhat involved.\(^4\) To consider the singularity of \( \beta(\omega, \phi_t) \), the linear magnetics case is assumed so that \( \phi_t \triangleq L_t i_t \) with \( f^{-1}(\phi_t) = \phi_t/L_t \). In this case, it is found that

\[ \beta(\omega, \phi_t) \triangleq \frac{K_m \phi_t}{JR_a} \left( \frac{2R}{L_t} - 2K_m \omega - \frac{K_m^2}{JR_a} \phi_t^2 \right) \]

The condition \( \beta(\omega, \phi_t) \neq 0 \) is satisfied for a region of the state space for which one would operate such a motor.\(^4\)

Remark

It is interesting to point out that if one considers speed control based on the first three equations (35), the system is not feedback linearizable.\(^4\) An alternative is input–output linearization control\(^4\) which is accomplished by applying the feedback

\[ V = \frac{(K_m^2/JR_a) \phi_t^2 \omega + u}{(K_m/JR_a) \phi_t} \]

to the system (36) to obtain

\[ \frac{d\phi_t}{dt} = -R f^{-1}(\phi_t) + K_m \phi_t \omega + \frac{JR_a}{K_m \phi_t} u \]
\[ \frac{d\omega}{dt} = u - \frac{\tau_t}{J} \] \hspace{1cm} (37)
\[ \frac{d\theta}{dt} = \omega \]
The controller is non-singular for $\phi_t > 0$ and the stability of the first equation in (37) (i.e. the zero dynamics) is determined by considering

$$ \frac{1}{2} \frac{d\phi_t^2}{dt} = -R\phi_t^{-1}(\phi_t) + K_m\phi_t^2\omega + \frac{JR_a}{K_m}u $$

$$ = -\left(\frac{R}{L}\phi_t^{-1}(\phi_t)\right) + \frac{JR_a}{K_m}u $$

where the second line assumes linear magnetics. It is then clear that if $\omega < \omega_{\text{max}} \triangleq \frac{R}{(K_mL_d)}$ that (38) is BIBO stable. Note that the adjustable resistor allows the designer to accommodate any specified maximum speed.

Though both feedback linearization and input–output linearization are applicable, the input–output linearization controller is favoured as the singularity condition is less stringent, the controller is simpler, and the stability of the zero dynamics is easily maintained.

5. DC–DC CONVERTERS

Developments in power electronics are the driving force behind the application of new control methods for electric motors. We complete the discussions of this paper by showing examples of similar control methods for power electronic DC–DC converters. A DC–DC converter is a power transfer device used to convert DC voltage from one level to another. For example, 120 V/60 Hz power from a wall socket is rectified to DC and then the voltage level shifted down by a converter to the level required by a personal computer. These systems are often modeled by second-order nonlinear control systems which are generically feedback linearizable. Although the presentation have is restricted to differential-geometric methods first given in Reference 49 passivity-based control methods have also been developed for DC–DC converters in Reference 50.

5.1. Boost converter

One type of DC–DC converter is the so-called boost converter shown in Figure 13. The equations describing this circuit are given by

$$ L \frac{di_L}{dt} = - (1 - d)V_C + V_{\text{in}} $$

$$ C \frac{dV_C}{dt} = (1 - d)i_L - V_C/R. $$

![Figure 13. DC–DC boost converter circuit](image-url)
The transformed system is controlled through the duty ratio input singular only if \( x \) of the capacitor voltage achieved indirectly using \( x_1 \). It is shown in Reference 50 that choosing \( v_0 = x_2 \) as the output for an input–output linearization controller results in unstable zero dynamics. On the other hand, choosing \( x_1 \) (inductor current) as output results in stable zero dynamics with control of the capacitor voltage achieved indirectly using \( x_1 \). However, the stability of the zero
dynamics is a local result. Both the input–output controller and the feedback linearization controller regulate the output capacitor indirectly. However, it appears that the feedback linearization controller may have an advantage in that its region of definition (i.e., where the singularities are absent) can be explicitly given in contrast to only a local guarantee of stability for the input–output controller.

5.2. Load estimator

The resistance $R$ models the load which is typically unknown. However, as $x_1$ and $x_2$ are measured, one can design a Krener–Isidori–Respondek type observer,\textsuperscript{45,14} i.e., a nonlinear observer with linear error dynamics, to estimate (the constant) $R$. With $x_3 \triangleq R$ and appending $dx_3/dt = 0$ to (40), a system model for the load estimation problem is

$$
\begin{align*}
\dot{x}_1 &= -\frac{1-u}{L} x_2 + \frac{V_{in}}{L} \\
\dot{x}_2 &= \frac{1-u}{C} x_1 - \frac{1}{C} x_2 \\
\dot{x}_3 &= 0
\end{align*}
$$

which is nonlinear in the unknown state variable $x_3$. To transform this to a new coordinate system where the dynamics are linear in $x_3$, consider the nonlinear transformation

$$
\begin{align*}
\tilde{x}_1 &= x_1 \\
\tilde{x}_2 &= C \ln(x_2) \\
\tilde{x}_3 &= 1/x_3
\end{align*}
$$

(41)

In these new co-ordinates, the system is described by

$$
d\tilde{x}/dt = A\tilde{x} + \phi(\tilde{x}, u) \\
y = \mathcal{C}\tilde{x}
$$

where

$$
\tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \\
\mathcal{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\
\phi(y, u) = \begin{bmatrix} -e^{x_2/C} (1-u)/L + V_{in}/L \\ (1-u)y_1 e^{-x_2/C} \\ 0 \end{bmatrix}
$$

The pair $\{\mathcal{C}, A\}$ is observable while $\phi(y, u)$ depends only on measured quantities so that it is straightforward to construct a nonlinear observer for $\tilde{x}_3$ whose error system is linear.
Remark

Similar to the above, a feedback linearization controller and load estimator can be designed for the buckboost converter.  

6. CONCLUDING REMARKS

The paper has surveyed a number of recent contributions to the control of electric motors using differential-geometric techniques. Such methods offer the benefit of a control system designed systematically and for global operation as opposed to using an ad hoc procedure and for local operation relying on gain scheduling. The potential advantages are reduced design time and improved performance for aggressive trajectories. The field of electric motors, because of well-understood dynamic models and widely available experimental test beds, also provides an excellent opportunity for the evaluation of the performance and robustness properties of nonlinear control methods, as well as for teaching research.

The paper considered a rich collection of examples including DC and AC motors, and some power electronic devices. Three types of nonlinearities were encountered: multiplicative, sinusoidal and saturation nonlinearities. While sinusoidal nonlinearities could be considered to be the most complex a priori, they are eliminated without difficulty using rotational transformations known as dq transformations. Multiplicative nonlinearities were actually more difficult to handle, especially when state variables were multiplied by inputs. Such nonlinearities were eliminated by appropriate state transformations and cancellation by feedback. Saturation nonlinearities present problems because they are usually known as tabular data. The paper showed that the theory was able to handle magnetic saturation, although the application to AC motors remains a subject of research.

Several methods, of differential-geometric approach were illustrated by the different motors, and sometimes by the same motor. In a few cases, feedback linearization was directly applicable. In others, the addition of an integrator to an input (as a state variable) allowed the theory to be applied. Feedback linearization is superior in that it provides for complete control of the dynamics of the system. However, a consistent observation is that the benefits of the method are significantly reduced if the variables to be controlled are not part of the transformed state variables. Input–output linearization does not suffer from this problem, as the output variables constitute the straight point for the derivation of the control algorithm. For input–output linearization, however, a significant issue is that of the stability of the dynamics that are made unobservable. In some cases, stability was found to be guaranteed because of some inherent property of the machine. On the other hand, for the induction motor, the hidden state variable was an angle which was not required to remain doubled. However, cases also were encountered where stability could neither be neglected nor guaranteed.

Nonlinear control methods for electric motors constitute a fertile ground for future research and development. The incorporation of magnetic saturation models in dynamic models and control laws is an important issue. From an industrial perspective, there is also an interest in methods for the self-tuning and adaptation of the control laws. Overall, there is a need to better understand the robustness properties of the methods to unmodelled dynamics (e.g. amplifier dynamics, hysteresis, etc.) and measurement noise (e.g. due to quantization and pulse-width modulation). For high performance, it is also typical to reach voltage and current limits. As was observed in this paper, the control of electric motors often reduces to a 2 × 2 problem with two voltages as
inputs, and a flux plus either the speed or the position as the outputs. Because of cross-couplings that are sometimes embedded in nonlinear transformations, hard saturation limits can pose problems not easily addressed by the theory.

Finally, it may be noted that electric motors constitute a target of application for nonlinear control methods in general, not just those based on the differential-geometric approach. Lyapunov methods, for example, are another source of recent contributions, including those relying on backstepping and on passivity theory.

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REFERENCES


